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Measures of Dirichlet Type on Regular Polygons and Their Moments

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The $(k-1)$-dimensional simplex is projected onto the convex hull of the kth roots of unity in $\mathbb{C}$, and a dihedral-group-invariant Dirichlet-type measure is thereby constructed. The integrals of monomials $z^m\bar{z}^n$ are obtained as single sums. A certain radial measure on the disc is obtained as a weak-*limit. © 1992 Academic Press, Inc.

By projecting a simplex onto a regular polygon we define a parametric family of measures and obtain formulas for the integrals of monomials. One of the main results can be stated as follows: choose $k=3, 4, 5, ...,$ $\alpha>0$, $a, b\in\mathbb{C}$ such that $|a|<\frac{1}{2}$ and $|b|<\frac{1}{2}$; let $\omega:=e^{2\pi i/k}$, $E_k:=\{(t_0, t_1, ..., t_{k-1})\in\mathbb{R}^k : t_j\geqslant 0 \text{ (each } j),\ \sum_{j=0}^{k-1} t_j=1\}$, then

$$\frac{\Gamma(k\alpha)}{\Gamma(\alpha)^k}\int_{E_k}\left(1-\sum_{j=0}^{k-1}(a\omega^j+b\omega^{-j})\,t_j\right)^{-k\alpha} \times (t_0t_1\cdots t_{k-1})^{\alpha-1}\,dt_0\,dt_1\cdots dt_{k-2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\alpha)_m(\alpha)_n(k(m+n+\alpha))_{2j}}{m!\,n!\,j!(k(m+n+\alpha)+1)_j}\,a^{km+j}b^{kn+j}.$$

In effect, we map E_k onto the unit k-gon X_k (the closed convex hull of the kth roots of unity) by the function $z:=\sum_{j=0}^{k-1} t_j\omega^j$. The stated formula immediately leads to a single-sum expression for the integral of $z^m\bar{z}^n$ $(m, n\in\mathbb{Z}_+)$. The method involves expansions related to the Lauricella F_D function and Chebyshev polynomials. We also show that the sequence of (normalized) measures, where $k\to\infty$ and $\alpha=\lambda/k$ for a fixed $\lambda>0$,

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converges weak-* to $(\lambda/\pi)\,(1-|z|^2)^{\lambda-1}\,dm_2(z)$ on the unit disc (m_2 is the Lebesgue measure on $\mathbb{R}^2 \cong \mathbb{C}$).

When k is even, there is a two-parameter version of the integral; the details are in Section 2. The author was led to studying these measures in the investigation of fractional integrals associated to orthogonal polynomials with dihedral symmetry [5]. Accordingly the measures in the present work have an invariance property for the dihedral group, which is generated by the rotation $z \mapsto z\omega$ and the reflection $z \mapsto \bar{z}$. Thus, instead of k parameters in the general Dirichlet distribution, there are just one or two parameters (as k is odd or even, respectively).

1. The One-Parameter Case

Choose $k=2, 3, 4, \ldots$ and $\alpha>0$. Define the probability measure on E_k (simplex in $\mathbb{R}^k$) by

$$d\mu_\alpha := \frac{\Gamma(k\alpha)}{\Gamma(\alpha)^k}\,(t_0 t_1 \cdots t_{k-1})^{\alpha-1}\,dt_0\,dt_1 \cdots dt_{k-2}$$

(this is a special case of the Dirichlet distribution: it can be interpreted as a measure in $\mathbb{R}^{k-1}$ with $t_{k-1} := 1-\sum_{j=0}^{k-2} t_j$). We use the same name for the measure induced on X_k by the equation

$$\int_{X_k} f(z)\,d\mu_\alpha(z) = \int_{E_k} f\left(\sum_{j=0}^{k-1} t_j \omega^j\right) d\mu_\alpha$$

(for continuous functions f on X_k). Note the invariance properties:

$$\int_{X_k} f(\omega z)\,d\mu_\alpha(z) = \int_{X_k} f(z)\,d\mu_\alpha(z) = \int_{X_k} f(\bar{z})\,d\mu_\alpha(z).$$

For $m=(m_0, \ldots, m_{k-1}) \in \mathbb{Z}_+^k$ let $|m| = \sum_{j=0}^{k-1} m_j$ and $m! := m_0!\,m_1! \cdots m_{k-1}!$, and if $x=(x_0, x_1, \ldots, x_{k-1}) \in \mathbb{C}^k$ let $x^m := x_0^{m_0} \cdots x_{k-1}^{m_{k-1}}$.

An iterated beta integral (e.g., Exton [6, p. 222]) shows that $\int_{E_k} t_0^{m_0} \cdots t_{k-1}^{m_{k-1}}\,d\mu_\alpha = \prod_{j=0}^{k-1} (\alpha)_{m_j}/(k\alpha)_{|m|}$, for $m \in \mathbb{Z}_+^k$. Now let $x \in \mathbb{C}^k$ with $|x_j|<1$ each j, and let $n \in \mathbb{Z}_+$, then

$$\int_{E_k} \left(\sum_{j=0}^{k-1} x_j t_j\right)^n d\mu_\alpha = \frac{n!}{(k\alpha)_n} \sum_{m \in \mathbb{Z}_+^k,\, |m|=n} \frac{(\alpha)_{m_0} \cdots (\alpha)_{m_{k-1}}}{m!}\, x^m$$

by the multinomial theorem. Multiply both sides by $(k\alpha)_n/n!$ and sum over $n \in \mathbb{Z}_+$ to obtain

$$\int_{E_k} \left(1 - \sum_{j=0}^{k-1} t_j x_j\right)^{-k\alpha} d\mu_\alpha = \prod_{j=0}^{k-1} (1 - x_j)^{-\alpha}$$

(this formula was used by Carlson [2] with k parameters to study Dirichlet averages of functions; see also his related work [1] on Lauricella's F_D function).

To obtain the desired integral, we let $x_j = a\omega^j + b\omega^{-j}$ and find a useful form for $\prod_{j=0}^{k-1} (1 - x_j)$ by means of Chebyshev polynomials. It is convenient to work with a homogeneous polynomial.

1.1. Definition. For $k = 1, 2, \ldots$, $\omega := e^{2\pi i/k}$ let $P_k(t, a, b) := \prod_{j=0}^{k-1} (t - (a\omega^j + b\omega^{-j}))$, $(t, a, b) \in \mathbb{C})$.

Since P_k is a polynomial we can restrict the variables by $0 < |ab| < |t|^2/4$ and introduce an auxiliary variable: $\xi := (t - \sqrt{t^2 - 4ab})/2$ (the principal branch of the square root). Thus $\xi(t - \xi) = ab$ and $|\xi| < 2|ab/t| < |t|/2$.

1.2. Proposition.

$$\begin{aligned} P_k(t, a, b) &= \sum_{j=0}^{[k/2]} (-1)^j \frac{k}{k-j} \binom{k-j}{j} t^{k-2j} a^j b^j - (a^k + b^k) \\ &= 2(ab)^{k/2} T_k(t/(2\sqrt{ab})) - (a^k + b^k) \\ &= (t - \xi)^k (1 - (a/(t - \xi))^k)(1 - (b/(t - \xi))^k) \end{aligned}$$

(*where T_k is the Chebyshev polynomial T_k* $(\cos\theta) = \cos k\theta$).

Proof. In the definition of P_k replace t by $\xi + ab/\xi$. Thus

$$\begin{aligned} P_k(t, a, b) &= \sum_{j=0}^{k-1} (\xi - a\omega^j - b\omega^{-j} + ab/\xi) \\ &= \prod_{j=0}^{k-1} [\xi^{-1}(\xi - a\omega^j)(\xi - b\omega^{-j})] \\ &= \xi^{-k}(\xi^k - a^k)(\xi^k - b^k) \\ &= (ab/\xi)^k (1 - (\xi/a)^k)(1 - (\xi/b)^k). \end{aligned}$$

This is the third part of the proposition because $ab/\xi = t - \xi$, $\xi/a = b/(t - \xi)$, and $\xi/b = a/(t - \xi)$. Also the product equals $\xi^k - (a^k + b^k) + (ab/\xi)^k$. The Chebyshev polynomials satisfy $T_k((z + z^{-1})/2) = (z^k + z^{-k})/2$ for arbitrary $z \in \mathbb{C}$, $z \neq 0$. Thus $\xi^k + (ab/\xi)^k = 2(ab)^{k/2} T_k((\xi + ab/\xi)/(2\sqrt{ab}))$ and $t = \xi +$

ab/ξ. A standard formula for T_k gives the first part (summation formula) of the proposition. (Note that $(k/(k-j))\binom{k-j}{j} = \binom{k-j+1}{j} - \binom{k-j-1}{j-2} \in \mathbb{Z}$.) ∎

At this point, we know

$$\int_{X_k} (1-(az+b\bar{z}))^{-k\alpha}\, d\mu_\alpha = P_k(1, a, b)^{-\alpha}.$$

Since $P_k(1, a, b)$ is an expression in ab and $a^k + b^k$ this reiterates the fact that $\int_{X_k} z^m \bar{z}^n\, d\mu_\alpha = 0$ unless $m \equiv n \bmod k$, which was already evident from the invariance properties of μ_α. However, $P_k(1, a, b)$ has $[k/2]+3$ terms and the direct expansion using the multinomial theorem would use $[k/2]+2$ summation variables (for a power series in a, b).

For $k=3$, we can obtain $\int_{X_3} z^{l+3s}\bar{z}^l\, d\mu_\alpha$ as a single sum immediately. Indeed, $P_3(1, a, b) = 1 - 3ab - (a^3+b^3)$, and

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(3\alpha)_{m+n}}{m!\, n!} a^m b^n \int_{X_3} z^m \bar{z}^n\, d\mu_\alpha$$
$$= P_3(1, a, b)^{-\alpha} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_{m+n+j}}{m!\, n!\, j!} 3^j a^{3m+j} b^{3n+j}.$$

Thus

$$\int_{X_3} z^{l+3s}\bar{z}^l\, d\mu_\alpha = \frac{(l+3s)!\, l!}{(3\alpha)_{2l+3s}} \sum_{m=0}^{[l/3]} \frac{(\alpha)_{l+s-m} 3^{l-3m}}{m!\, (m+s)!\, (l-3m)!}.$$

In this case, the measure μ_1 is exactly the (normalized) area measure on the triangle X_3 (since the simplex E_3 is two-dimensional). This integral was studied (but not found explicitly) in [3]. A recurrence formula for the integral and some families of orthogonal polynomials were obtained (note that the variable α in that paper corresponds to $\alpha - 1$ in this one).

To make further progress on the expansion of $P_k(1, a, b)^{-\alpha}$ we use the triple product formula in Proposition 1.2. Specialize to $t=1$ and let $\zeta := (1-\sqrt{1-4ab})/2$ (a special value of ξ).

1.3. Proposition. *$P_k(1, a, b) = (1-\zeta)^k(1-(a/(1-\zeta))^k)(1-(b/(1-\zeta))^k)$, and $|\zeta| < 2|ab|$, for $|ab| < \frac{1}{4}$.*

A quadratic transformation for hypergeometric series gives the power series expansion for $(1-\zeta)^{-\gamma}$ in terms of ab.

1.4. Lemma. *For $|ab| < \frac{1}{4}$, $\gamma \neq -1, -2, -3, \ldots$,*

$$(1-\zeta)^{-\gamma} = \sum_{j=0}^{\infty} \frac{(\gamma)_{2j}}{(\gamma+1)_j\, j!} (ab)^j.$$

Proof. Start with the standard quadratic transformation (see, e.g., Gasper and Rahman [7, p. 60 (3.17)])

$$ {}_2F_1(2c, 2d; c+d+1/2; \zeta) = {}_2F_1(c, d; c+d+1/2; 4\zeta(1-\zeta)) $$

(to prove this expand the right side with the binomial theorem and use Saalschütz's summation formula), and let $c = \gamma/2$, $d = (\gamma+1)/2$ (so $c+d+1/2 = 2d$). Then

$$ (1-\zeta)^{-\gamma} = {}_2F_1(\gamma/2, (\gamma+1)/2; \gamma+1; 4\zeta(1-\zeta)) $$
$$ = \sum_{j=0}^{\infty} \frac{(\gamma)_{2j}}{(\gamma+1)_j j!} (\zeta(1-\zeta))^j, $$

since

$$ (\gamma/2)_j ((\gamma+1)/2)_j 2^{2j} = (\gamma)_{2j}. \quad \blacksquare $$

1.5. THEOREM. *For* $|a|, |b| < \frac{1}{2}$,

$$ \int_{X_k} (1-(az+b\bar{z}))^{-k\alpha} \, d\mu_\alpha $$
$$ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha)_m (\alpha)_n (k(m+n+\alpha))_{2j}}{m! \, n! \, j! (k(m+n+\alpha)+1)_j} a^{km+j} b^{kn+j}. $$

Proof. The integral equals

$$ P_k(1, a, b)^{-\alpha} = (1-\zeta)^{-k\alpha} (1-(a/(1-\zeta))^k)^{-\alpha} (1-(b/(1-\zeta))^k)^{-\alpha} $$

(by Proposition (1.3))

$$ = (1-\zeta)^{-k\alpha} \sum_{m=0}^{\infty} \frac{(\alpha)_m}{m!} a^{km} (1-\zeta)^{-km} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} b^{kn} (1-\zeta)^{-kn}. $$

Now rearrange the sum and expand $(1-\zeta)^{-k(m+n+\alpha)}$ by Lemma 1.4. $\blacksquare$

We use the theorem to find the nonzero moments of the measure μ_α.

1.6. THEOREM. *For* $s, l \in \mathbb{Z}_+$,

$$ \int_{X_k} z^{ks+l} \bar{z}^l \, d\mu_\alpha $$
$$ = \int_{X_k} z^l \bar{z}^{ks+l} \, d\mu_\alpha = (ks+l)! \, l! $$
$$ \times \sum_{n=0}^{[l/k]} \frac{(\alpha)_{n+s} (\alpha)_n}{(n+s)! \, n! \, (l-kn)! \, (k\alpha)_{k(2n+s)} (k(2n+s+\alpha)+1)_{l-kn}}. $$

Proof. The desired integral equals $l!(ks+l)!/(k\alpha)_{ks+2l}$ times the coefficient of $a^{ks+l}b^l$ in the triple sum; thus $j=l-kn$, $m=n+s$. Note that $(\alpha k)_{ks+2l}=(\alpha k)_{k(2n+s)}(k(\alpha+2n+s))_{2(l-kn)}$. ∎

In the special case $0\leqslant l<k$, this integral simply equals $(ks+l)!(\alpha+1)_s/s!(1+k\alpha)_{ks+l}$.

We compare the measure μ_α on X_k with the area measure m_2 (there does not seem to be a tractable measure of Dirichlet (beta) type on X_k for $k>3$). By the methods of [4] where area measure was used on X_6 for practical reasons,

$$\left(\frac{k}{2}\sin\frac{2\pi}{k}\right)^{-1}\int_{X_k} z^{ks+l}\bar{z}^l\,dm_2(z)$$
$$=\frac{2}{(ks+2l+1)_2}\,{}_3F_2\left(\begin{matrix}-l,\,-ks-l,\,1\\ -l-ks/2,\,-l-ks/2+1/2\end{matrix};\ \cos^2\frac{\pi}{k}\right).$$

Except for the crystallographic values $k=3$, 4, 6, the argument $\cos^2(\pi/k)$ is irrational. There is no convenient relationship to the measures μ_α on X_k. On the other hand μ_α agrees (as a linear functional) on polynomials of degree $<k$ with a certain measure on the unit disc. This measure is in the parametric family associated to the disc polynomials (an orthogonal family, see Koornwinder [9, pp. 448–449]).

Indeed, for $\lambda>0$, define

$$d\nu_\lambda(z):=\frac{\lambda}{\pi}(1-|z|^2)^{\lambda-1}\,dm_2(z),$$

a probability measure on the unit disc $D:=\{z\in\mathbb{C}: |z|\leqslant 1\}$. Further $\int_D z^m\bar{z}^n\,d\nu_\lambda(z)=\delta_{mn}(m!/(\lambda+1)_m)$ (by use of polar coordinates and the beta integral). For $|a|$, $|b|<\frac{1}{2}$ we have that

$$\int_D(1-(az+b\bar{z}))^{-\lambda}\,d\nu_\lambda(z)$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda)_{m+n}}{m!\,n!}a^m b^n\,\delta_{mn}\frac{m!}{(\lambda+1)_m}$$
$$=\sum_{m=0}^{\infty}\frac{(\lambda)_{2m}}{(\lambda+1)_m m!}(ab)^m=(1-\zeta)^{-\lambda}$$

(by Lemma 1.4). This shows that $\nu_{k\alpha}$ has the same moments (integrals of $z^m\bar{z}^n$) as μ_α up to degree $k-1$.

1.7. PROPOSITION. *Fix $\lambda > 0$ and let $\alpha_3, \alpha_4, \alpha_5, \ldots$ be a sequence of positive numbers such that $k\alpha_k \to \lambda$; then the sequence μ_{α_k} (measure on X_k) converges weak-* to ν_λ (in the dual space of $C(D)$).*

Proof. For any polynomial $p(z, \bar{z})$,

$$\int_{X_k} p(z, \bar{z})\, d\mu_{\alpha_k} \to \int_D p(z, \bar{z})\, d\nu_\lambda(z)$$

(since for all large k, $k > \deg p$). Polynomials are uniformly dense in $C(D)$ and the sequence of measures is bounded in norm. ∎

This is a special case of a result of Jiang [8] who found weak-* limits of general Dirichlet measures on polygons in the unit circle subject to certain regularity conditions on the parameters. The proof depends on characteristic functions, rather than the explicit polynomial formulas used here.

2. THE TWO-PARAMETER CASE

When k is even the rotation group of X_k has a subgroup of index 2 and thus allows the introduction of another parameter while maintaining some invariance properties.

Let $k = 2h$, $h = 1, 2, \ldots$; $\omega = e^{2\pi i/k} = e^{\pi i/h}$. Choose parameters $\alpha, \beta > 0$ and define

$$d\mu_{\alpha\beta} = \frac{\Gamma(h(\alpha + b))}{\Gamma(\alpha)^h \Gamma(\beta)^h} (t_0 t_2 \cdots t_{k-2})^{\alpha - 1} (t_1 t_3 \cdots t_{k-1})^{\beta - 1}\, dt_0\, dt_1 \cdots dt_{k-2},$$

a measure on E_k, or X_k (as in Section 1). If f is a continuous function on X_k, the following hold:

$$\int_{X_k} f(z)\, d\mu_{\alpha\beta} = \int_{X_k} f(\bar{z})\, d\mu_{\alpha\beta} = \int_{X_k} f(z\omega^2)\, d\mu_{\alpha\beta} = \int_{X_k} f(z\omega)\, d\mu_{\beta\alpha}.$$

By the same methods as in Section 1 we claim that

$$\int_{E_k} \left(1 - \sum_{j=0}^{k-1} t_j x_j\right)^{-h(\alpha+\beta)} d\mu_{\alpha\beta} = \prod_{j=0}^{h-1} ((1 - x_{2j})^{-\alpha} (1 - x_{2j+1})^{-\beta})$$

(where $|x_j| < 1$ each j). For numbers a, b with $|a|$, $|b| < \frac{1}{2}$, and $x_j = a\omega^j + b\omega^{-j}$ as before, $\prod_{j=0}^{h-1} (1 - x_{2j}) = P_h(1, a, b)$ and $\prod_{j=0}^{h-1} (1 - x_{2j+1}) = P_h(1, a\omega, b\bar{\omega})$. Of course, $P_k(1, a, b) = P_h(1, a, b)\, P_h(1, a\omega, b\bar{\omega})$, which shows how the integral formula collapses to the one-parameter case when $\alpha=\beta$. The polynomial

$$\begin{aligned} P_h(1, a\omega, b\bar{\omega}) &= 2(ab)^{h/2}T_h(1/\sqrt{ab})) + (a^h+b^h) \\ &= (1-\zeta)^h(1+(a/(1-\zeta))^h)(1+(b/(1-\zeta))^h) \end{aligned}$$

by the same trick as in 1.3, a slight difference from $P_h(1, a, b)$. The extra parameter necessitates more summation variables which possibly excuses some more notation.

Already in the trivial case $k=2$ (where $z=2t_0-1$)

$$\int_{X_2} z^n \, d\mu_{\alpha\beta} = \frac{\Gamma(a+b)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 (2t_0-1)^n \, t_0^{\alpha-1}(1-t_0)^{\beta-1} \, dt_0 = \gamma_n(\alpha, \beta),$$

where

$$\gamma_n(\alpha, \beta) := \frac{n!}{(\alpha+\beta)_n} \sum_{j=0}^{n} \frac{(\alpha)_{n-j}(\beta)_j}{(n-j)!\, j!} (-1)^j,$$

essentially a general terminating ${}_2F_1$ evaluated at -1. We collect some properties of $\gamma_n(\alpha, \beta)$.

2.1. Lemma. *For $\alpha \geqslant 0$, $\beta \geqslant 0$, $\alpha+\beta>0$, and $n \in \mathbb{Z}_+$,*

(i) $(1-t)^{-\alpha}(1+t)^{-\beta} = \sum_{n=0}^{\infty} ((\alpha+\beta)_n/n!)\, \gamma_n(\alpha, \beta)\, t^n$ *for* $|t|<1$;

(ii) $\gamma_n(\beta, \alpha) = (-1)^n \gamma_n(\alpha, \beta)$;

(iii) $|\gamma_n(\alpha, \beta)| \leqslant 1$;

(iv) $\gamma_{2n}(\alpha, \beta) > 0$;

(v) $\operatorname{sgn} \gamma_{2n+1}(\alpha, \beta) = \operatorname{sgn}(\alpha-\beta)$.

Proof. The generating function (i) and properties (ii) and (iii) are obvious (recall $\sum_{j=0}^{n} (\alpha)_{n-j}(\beta)_j/(n-j)!\, j! = (\alpha+\beta)/n!$, the Chu–Vandermonde sum). By (ii) it suffices to consider $\alpha \geqslant \beta$. In this case

$$\begin{aligned} (1-t)^{-\alpha}(1+t)^{-\beta} &= (1-t^2)^{-\beta}(1-t)^{-(\alpha-\beta)} \\ &= \sum_{m=0}^{\infty} \frac{(\beta)_m}{m!} \sum_{j=0}^{\infty} \frac{(\alpha-\beta)_j}{j!} t^j, \end{aligned}$$

which shows that $\gamma_n(\alpha, \beta) = (n!/(\alpha+\beta)_n) \sum_{m=0}^{[n/2]} (\beta)_m(\alpha-\beta)_{n-2m}/m!\,(n-2m)!$. This shows $\gamma_n(\alpha, \beta)>0$ for $n=0, 1, 2, ..., \alpha \geqslant \beta$, except $\gamma_{2n+1}(\alpha, \alpha)=0$. ∎

2.2. THEOREM. *For* $|a|, |b| < \frac{1}{2}$,

$$\int_{X_k} (1-(az+b\bar{z}))^{-h(\alpha+\beta)}\, d\mu_{\alpha\beta}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{\begin{pmatrix} (\alpha+\beta)_m(\alpha+\beta)_n \gamma_m(\alpha,\beta) \\ \times \gamma_n(\alpha,\beta)(h(m+n+\alpha+\beta))_{2j} \end{pmatrix}}{m!\, n!\, j!\, (h(m+n+\alpha+\beta)+1)_j} a^{hm+j} b^{hn+j}.$$

Proof. As in 1.5, the integral equals

$$\begin{aligned} &P_h(1,a,b)^{-\alpha} P_h(1, a\omega, b\bar{\omega})^{-\beta} \\ &\quad = (1-\zeta)^{-h(\alpha+\beta)}(1-(a/(1-\zeta))^h)^{-\alpha}(1-(b/(1-\zeta))^h)^{-\alpha} \\ &\quad\quad \times (1+(a/(1-\zeta))^h)^{-\beta}(1+(b/(1-\zeta))^h)^{-\beta} \\ &\quad = (1-\zeta)^{-h(\alpha+\beta)} \sum_{m=0}^{\infty} \frac{(\alpha+\beta)_m}{m!} \gamma_m(\alpha,\beta)\, a^{hm}(1-\zeta)^{-hm} \\ &\quad\quad \times \sum_{n=0}^{\infty} \frac{(\alpha+\beta)_n}{n!} \gamma_n(\alpha,\beta)\, b^{hn}(1-\zeta)^{-hn} \end{aligned}$$

(by use of Lemma 2.1(i)). Now expand $(1-\zeta)^{-h(\alpha+\beta+m+n)}$ by Lemma 1.4. ∎

Hence we can find the nonzero moments of $\mu_{\alpha\beta}$.

2.3. COROLLARY. *For* $s, l \in \mathbb{Z}_+$,

$$\int_{X_k} z^{hs+l}\bar{z}^l\, d\mu_{\alpha\beta} = \int_{X_k} z^l \bar{z}^{hs+l}\, d\mu_{\alpha\beta} = (-1)^s \int_{X_k} z^{hs+l}\bar{z}^l\, d\mu_{\beta\alpha}$$

$$= (hs+l)!\, l! \sum_{n=0}^{[l/h]} \frac{(\alpha+\beta)_{n+s}(\alpha+\beta)_n\, \gamma_{n+s}(\alpha,\beta)\, \gamma_n(\alpha,\beta)}{\begin{pmatrix} (n+s)!\, n!(l-hn)!(h(\alpha+\beta))_{h(2n+s)} \\ \times (h(\alpha+\beta+2n+s)+1)_{l-hn} \end{pmatrix}}.$$

The integral is positive if s is even, while if s is odd it has the same sign as $\alpha-\beta$ $(0$ *if* $\alpha=\beta)$.

Proof. The formula follows from the theorem (same as 1.6). If $\alpha \neq \beta$, $\operatorname{sgn}(\gamma_{n+s}(\alpha,\beta)\,\gamma_n(\alpha,\beta)) = (\operatorname{sgn}(\alpha-\beta))^s$, thus each term in the sum is of the same sign. ∎

2.4. COROLLARY. *The measure* $\mu_{\alpha\beta}$ *on* $X_k \subset D$ *(the disc) has the same moments as* $\nu_{h(\alpha+\beta)}$ *up to degree* $h-1$.

Proof. The only nonzero moments in this range are

$$\int_{X_k} z^l \bar{z}^l \, d\mu_{\alpha\beta} = \frac{l!}{(h(\alpha+\beta)+1)_l} \qquad \text{for} \quad 2l < h. \quad \blacksquare$$

As in the one-parameter case, fix $\lambda > 0$ and choose a sequence of pairs of positive numbers (α_h, β_h) such that $h(\alpha_h + \beta_h) \to \lambda$ as $h \to \infty$. Then the measures $\mu_{\alpha_h \beta_h}$ on X_{2h} converge weak-* to ν_λ as $h \to \infty$. This has the same proof as 1.7.

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